Emergent Behavior of a Structured Vacuum

Justin J. Meza
justin.meza@gmail.com

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Abstract

We model the vacuum as a structured medium with three fields: A displacement (shear) u^i, an orthonormal frame $e^a{}_\mu$ encoding rotations (twist), and a scalar (scale) field σ. Using a constructive derivation that removes non-physical couplings, we demonstrate that (i) the shear-only sector reproduces the Einstein–Hilbert [2, 3] action, (ii) the twist-only sector yields Maxwell electrodynamics [4], and (iii) the scale sector produces a Bergmann–Wagoner [5, 6, 7] scalar-tensor theory. While a structured vacuum is able to recover familiar physics at effective limits, is also has the potential to support additional medium-induced interactions, including contributions to vacuum birefringence and light-by-light scattering [21], and a spin–quadrupole channel for gravitational radiation. We release an open-source simulator implementing the model's dynamics to facilitate reproducibility and to enable exploration of cross-couplings and testable consequences.

1 Why Revisit Empty Space?

In the nineteenth century, physicists pictured an *ether*—a rigid, all-pervasive substrate that conveyed light yet remained otherwise undisturbed by matter. With special relativity, twentieth century physicists dismissed the ether's rest frame, and quantum field theory recast the vacuum as the state nullified by all annihilation operators. While these opposing views of the vacuum helped aid the explanation of their respective physical theories, they both suffer from a complementary deficiency: The ether carries *too much* structure (an absolute rest frame), while the quantum vacuum carries *too little*.

In this paper, we chart a course for a middle path: A *structured vacuum* that is material enough to endow space-time with elasticity and spin, yet symmetric enough to satisfy the relativity principle. We argue that observable phenomena arise as emergent behavior governed by the rules of interaction among the medium that comprises the structured vacuum and we show that our framework is flexible enough to encompass a variety of prior theories in the appropriate limits. We also release as open source a simulator that models a discrete version of our framework, which can be used to reproduce our results and as a test-bed for exploring structured vacuum dynamics.

2 Statement of Principles

We begin by establishing two principles that will guide our search for the dynamics that govern the structured vacuum. We develop the rest of our structured vacuum from successive derivation from these first principles.

Relativity of Inertial Motion

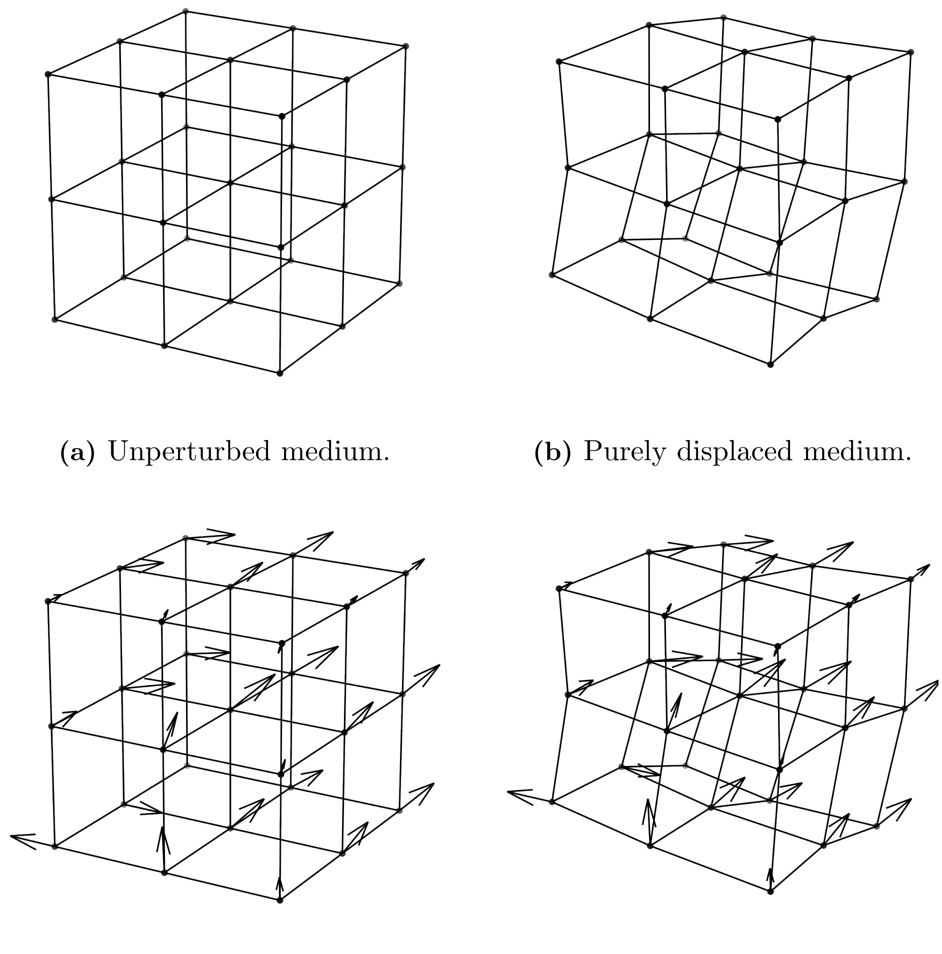
> *The differential equations governing the vacuum medium retain the same form in every inertial coordinate system.*

No experiment confined to a moving laboratory can reveal its state of motion with respect to the medium. Specifically, the speed of small amplitude elastic or torsional waves that propagate through the medium are isotropic in every inertial frame.

Universality of the Medium

> *All physical fields are collective excitations of a single continuous medium whose local state is described by*
>
> $$\left[u^i(x),\ e^a{}_\mu(x)\right],$$
>
> *where u^i encodes translations and $e^a{}_\mu$ encodes intrinsic rotations.*

The vacuum combines both elastic translations and torsional rotations in a single substrate. This primitive geometry naturally gives rise to both compressional waves as well as rotational waves without selecting a preferred rest frame. Figure 1 shows a visual representation of some examples of perturbations the structured vacuum medium supports.



(a) Unperturbed medium. **(b)** Purely displaced medium.

(c) Purely rotated medium. **(d)** Combined perturbation medium.

Figure 1: Various forms of structured vacuum perturbation at a fixed time, t.

Building upon these two principles as a foundation, our goals for the rest of this paper are to (a) find the most economical set of relativistically covariant differential equations obeyed by $(u^i,\ e^a{}_\mu)$ and (b) to compare their consequences with observation. Section 3 sets the stage by defining the kinematics of the structured vacuum's inertial system; Section 4 shows how at two derivatives the linearized Einstein equation, the vacuum Maxwell equations [4], and a Bergmann-Wagoner [5, 6, 7] scalar-tensor theory of gravity emerge; and Section 5 shows how at four derivatives the model adds additional precision to the prior theories with the Einstein–Hilbert [2, 3] action of general relativity and the non-linear Maxwell equations emerging along with other potential cross-terms that indicate the potential for the medium to support vacuum birefringence, light-by-light scattering [21], and spin–quadrupole gravitational radiation. Section 6 describes the design, implementation, and usage of our open source simulator.

3 Vacuum Geometry and Kinematics

We work in an inertial system $\mathcal{K}$ with coordinates

$$x^\mu = (t, x, y, z).$$

We use Greek letters to refer to *external* space-time indices. Material points in the structured vacuum carry labels $\xi^i = (\xi, \eta, \zeta)$ and are fixed (Lagrangian coordinates); their world-lines are the maps

$$Y^\mu : \mathbb{R}^3 \times \mathbb{R} \;\to\; \mathbb{R}^4, \qquad (\xi^i, t) \mapsto Y^\mu(\xi^i, t).$$

We raise and lower world indices with $g_{\mu\nu}$ and internal frame indices with η_{ab}:

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \, \eta_{ab}, \quad e_{a\nu} = \eta_{ab} e^b{}_\nu, \quad e^{a\mu} = g^{\mu\nu} e^a{}_\nu.$$

For Lorentz matrices we use

$$\Lambda_\mu{}^\nu = \eta_{\mu\rho} \, \Lambda^\rho{}_\sigma \, \eta^{\sigma\nu}, \qquad (\Lambda^{-1})^\mu{}_\nu = \eta^{\mu\rho} \Lambda^\sigma{}_\rho \eta_{\sigma\nu}.$$

Displacements It is useful to split the world-line into the identity embedding plus a small deviation:

$$Y^\mu(\xi, t) \;=\; Y_0^\mu(\xi, t) + u^\mu(\xi, t), \qquad Y_0^\mu(\xi, t) \;=\; (t, \xi, \eta, \zeta),$$

so that the spatial components (u^x, u^y, u^z) measure the *displacement* of the labelled point ξ^i from the reference rest frame.

Rotations A local tetrad $e^a{}_\mu(\xi, t)$ attaches an orthonormal basis to each point to record intrinsic rotations relative to its neighboring points,

$$e^a{}_\mu(\xi, t), \qquad e^a{}_\mu \, e_{a\nu} = g_{\mu\nu}, \; e^a{}_\mu \, e^{b\mu} = \eta^{ab}.$$

We use Latin letters to refer to the *internal* material frame indices. Each e^a records how the element is *rotated* relative to its neighbours. We use the set of material frames attached to each point to form the metric for the observable geometry, $g_{\mu\nu}$, and ideal flat-space, η^{ab}.

As a matter of bookkeeping for this rotation-based field, we define a translation between local Lorenz gauge and teleparallel gauge. Let $\omega^a{}_{b\mu}$ denote a flat spin connection [9, 10], such as,

$$R^a{}_{b\mu\nu}\,[\omega] \; = \; 0.$$

Here, ω is pure gauge and carries no new dynamics; thus, we have

$$\omega^a{}_{b\mu} \; = \; \left(\Lambda^{-1}\partial_\mu \Lambda\right)^a{}_b,$$

for some Lorentz matrix $\Lambda(x)$. We define,

$$\mathcal{D}_\mu e^a{}_\nu \equiv \partial_\mu e^a{}_\nu + \omega^a{}_{b\mu} e^b{}_\nu - \Gamma^\rho{}_{\mu\nu} e^a{}_\rho.$$

Without loss of generality—and motivated by our desire to keep our notation tidy—we evaluate covariant derivatives in the inertial gauge, $\omega^a_{b\mu} = 0$. At any time, we can restore local Lorentz covariance with $e^a{}_\mu \mapsto \Lambda^a{}_b(x) e^b{}_\mu(x)$ and the standard transformation $\omega^a{}_{b\mu} \mapsto \Lambda^a{}_c(x)\omega^c{}_{d\mu}\left(\Lambda^{-1}\right)^d{}_b + \Lambda^a{}_c \partial_\mu \left(\Lambda^{-1}\right)^c_b$.

Dilations The determinant of the tetrad,

$$e \equiv \det(e^a{}_\mu), \qquad \sqrt{-g} = |e|.$$

Under $x'^\mu = \Lambda^\mu{}_\nu x^\nu$,

$$e'(x') = \det\!\left(\Lambda^{-1}\right) e(x), \qquad |e|'(x') = |\det\!\left(\Lambda^{-1}\right)|\,|e|(x).$$

For proper orthochronous Lorentz transformations, $|\det(\Lambda^{-1})| = 1$. As a scalar density of weight $+1$, $|e|$ obeys

$$|e|'(x') = \left|\det \frac{\partial x}{\partial x'}\right| \, |e|(x).$$

We define the dimensionless dilaton

$$\sigma(x) \equiv \ln \frac{|e|(x)}{|e|_0},$$

where $|e|_0$ is a fixed reference density of the same weight; then σ is a true scalar and $\nabla_\mu \sigma$ a true covector.

Lorentz Boosts As an example, let $\mathcal{K}'$ move with constant velocity v along the x-axis of $\mathcal{K}$. We set $c = 1$ so that time and length share the same units. The standard Lorentz matrix is

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \frac{v}{c}, \ \ \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

The principle of relativity requires that u^μ transforms as a contravariant four-vector in every inertial frame:

$$\begin{aligned} &u'^\mu(\xi,t) \ = \ \Lambda^\mu{}_\nu \, u^\nu(\xi,t), \qquad u'^0 \ = \ -\gamma\beta\, u^x, \\ &u'^x = \gamma\, u^x, \quad u'^y = u^y, \quad u'^z = u^z \quad (\text{with } u^0 = 0 \text{ in } \mathcal{K}). \end{aligned} \tag{1}$$

Carrying out the matrix multiplication, we get

$$u'^0 = \gamma\left(u^0 - \beta\, u^x\right), \qquad u'^x = \gamma\left(u^x - \beta\, u^0\right), \qquad u'^y = u^y, \ u'^z = u^z.$$

Since the time component $u^0 = 0$ in $\mathcal{K}$ by construction, we find that

$$u'^0 = -\gamma\beta\, u^x, \qquad u'^x = \gamma\, u^x, \qquad u'^y = u^y, \ u'^z = u^z.$$

Thus, time-like components appear in $\mathcal{K}'$ even if absent in $\mathcal{K}$; they encode the relativity of simultaneity for the stretched vacuum medium. If u^i describes the elastic "stretch" of a material measured at one instant, the boosted observer sees that different parts of the medium have that stretch at different times. The non-zero time component u'^0 is the bookkeeping term that encodes *when* each part of the stretch happens in $\mathcal{K}'$.

Similarly, take the tetrad $e^a{}_\mu$ in $\mathcal{K}$: Under a boost $\Lambda^\mu{}_\nu$ between $\mathcal{K}$ and $\mathcal{K}'$ the transformation law reads

$$e'^a{}_\mu(x') = (\Lambda^{-1})^\nu{}_\mu \, e^a{}_\nu(x), \qquad e'^a{}_\mu e'_{a\nu} = g'_{\mu\nu}, \quad g'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma}. \tag{2}$$

Only the external space-time slot ν transforms while the internal material frame index a remains inert. Intrinsic orientation belongs to the medium itself, not to the observer. In other words, if the medium is "stretched" in a certain internal direction, every observer must agree which internal axis a is stretched, but they will disagree on how the axis is decomposed into external space-time coordinates.

Finally, take the (signed) determinant of the tetrad under a boost Λ:

$$e' \;=\; \det\!\big((\Lambda^{-1})^{\nu}{}_{\mu}\, e^{a}{}_{\nu}\big) \;=\; \det\!\big(\Lambda^{-1}\big)\, \det(e^{a}{}_{\nu}) \;=\; \det\!\big(\Lambda^{-1}\big)\, e. \tag{3}$$

For a proper orthochronous Lorentz transformation we have $\det(\Lambda) = +1$, hence $e' = e$ and therefore $|e'| = |e|$. If parity or time orientation is flipped ($\det \Lambda = -1$), then $e' = -e$ while $|e'| = |e|$. Under global Lorentz coordinate transformations the Jacobian has unit determinant, so $|e|$ is numerically unchanged; under general coordinate changes it transforms as a density as stated above.

The set of rules in Equations (1), (2), and (3) provide us with the complete Lorentz behavior we need to build invariant actions. In the next section, we identify the only set of non-trivial differential equations that obey the principles of relativity and universality.

4 Quadratic Vacuum Dynamics

To give the reader a feel for the way the vacuum is structured, we start with a simplified quadratic action. We build this action from the first principles we define in Section 2 and argue why the geometry of the vacuum *must* give rise to the terms in the action. In Section 5, we use this same process to derive the full-blown action (at least for the present work) at the quartic order.

The displacement gradient $\partial_\mu u_\nu$ splits into two parts: a symmetric part representing strain and an antisymmetric part representing rigid rotation. To linear order, only the symmetric part changes lengths. We therefore write the effective metric as the pullback of the ambient Minkowski metric by the embedding,

$$g_{\mu\nu} = \partial_\mu Y^\alpha\, \partial_\nu Y_\alpha,$$

where for small displacements, we write

$$Y_0^\alpha(\xi, t) \;=\; (t, \xi, \eta, \zeta), \qquad Y^\alpha = Y_0^\alpha + u^\alpha, \qquad g_{\mu\nu} = \eta_{\mu\nu} + 2\,\partial_{(\mu} u_{\nu)} + \mathcal{O}\big((\partial u)^2\big).$$

From here on we work in the Eulerian description, identifying $Y^\alpha(x) = x^\alpha + u^\alpha(x)$ near the identity so that $\partial_\mu \equiv \partial/\partial x^\mu$.

The antisymmetric derivative of the tetrad, $\partial_{[\mu} e^a{}_{\nu]}$, isolates the piece of the action that cannot be absorbed into a smooth metric change, and supplies torsion [8],

$$T^a{}_{\mu\nu} = 2\,\mathcal{D}_{[\mu} e^a{}_{\nu]}.$$

At the ground state, $u^\mu = 0$ and $g_{\mu\nu}$ reduces to the flat background, $\eta_{\mu\nu}$; similarly, when $e^a{}_\mu = \delta^a_\mu$ then $T^a{}_{\mu\nu} = 0$, and no torsion is sourced. Since both metric perturbations ($h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$) and torsion ($T^a{}_{\mu\nu}$) vanish in equilibrium, expanding their action to second order in derivatives suffices to capture low-energy waves and their leading interactions, which we pursue next.[1]

4.1 Quadratic Action

Our initial goal is to write an effective field theory for long-wavelength, small-amplitude excitations of a continuous structured vacuum medium. We identify three necessary and sufficient criteria to guide our search:

1. **Locality.** We enforce that the energy stored at any space-time point x depends only on the fields *at* x and on a *finite* number of derivatives evaluated *at* x. The energy at x *may not* depend on the values of the fields at some distant point $y \neq x$. This ensures that changes in a far away field do not instantaneously change forces over all space, which would contradict both prior experimental evidence and relativity.

2. **Covariance.** We require that the laws that govern the system look *the same* to every inertial observer. This is meant to mimic the Lorentz symmetry that we observe empirically in nature. Practically speaking, this implies that all pieces of the action must be constructed from tensors whose covariant combination keeps the same algebraic arrangement of indices.

3. **Scalarity.** We stipulate that after we contract all Greek (space-time) and Latin (internal) indices, the integrand is a scalar quantity. Integrating that scalar over space-time gives a single real number: the

[1] We will examine non-linear couplings between these excitations in subsequent sections.

action S. A scalar has the same numerical value in any coordinate system, ensuring the variation $\delta S = 0$ leads to equations of motion that do not depend on the observer's inertial frame or the choice of internal basis of the medium. Because the action is a scalar, the derived field equations carry the correct form to covariantly commute from one space-time inertial frame—or one internal basis—to another.

To satisfy locality, we leverage the local, Lorentz-covariant nature of the independent degrees of freedom of the structured vacuum u^μ, $e^a{}_\mu$, and $|e|$ that we demonstrated in Equations (1), (2), and (3). These quantities become the only admissible building blocks for our action. From these, we define the following first-derivative tensors:

$$S_{\mu\nu} \equiv \nabla_{(\mu} u_{\nu)}, \qquad T^a{}_{\mu\nu} \equiv 2\,\mathcal{D}_{[\mu} e^a{}_{\nu]}, \qquad \nabla_\mu \sigma,$$

In the teleparallel (inertial) gauge with $\omega^a{}_{b\mu} = 0$ and Levi–Civita Γ, we have $T^a{}_{\mu\nu} = 2\,\partial_{[\mu} e^a{}_{\nu]}$. Starting from these first-derivative tensors, we then seek all quadratic derivatives that satisfy our criteria of locality, covariance, and scalarity. This leaves us with only the following invariants.[2]

Shear Invariant $S_{\mu\nu}S^{\mu\nu}$. Contracting the symmetric strain tensor with itself gives us

$$S_{\mu\nu}S^{\mu\nu} \;=\; \nabla_{(\mu} u_{\nu)}\, \nabla^{(\mu} u^{\nu)} \;=\; (\partial u)^2 \quad \text{(to linear order).}$$

Each of the two S terms contributes one derivative and the Greek indices are fully contracted; since no internal (Latin) index is involved, $(\partial u)^2$ satisfies our criteria. [3]

Twist Invariant $T^a{}_{\mu\nu}T_a{}^{\mu\nu}$. With torsion we must also contract the internal index a. Working in the teleparallel (inertial) gauge $\omega = 0$ and with Levi–Civita Γ, we start by expanding:

$$T^a{}_{\mu\nu} \;\equiv\; 2\,\partial_{[\mu} e^a{}_{\nu]} \;=\; \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu, \tag{4}$$

[2]For a listing of the candidates that do not satisfy these criteria, see Appendix A.

[3]Though u and e are independent fields, because they both induce the same metric, $g_{\mu\nu}$, their volume pieces obey $\sigma = S^\mu{}_\mu + \mathcal{O}\big((\partial u)^2\big)$. The missing $(S^\mu{}_\mu)^2$ term is a deliberate choice: we keep isotropic dilation in the scale field σ and keep only its gradient term at quadratic order.

and then contracting, substituting, and distributing:

$$\begin{aligned} T^a{}_{\mu\nu} T_a{}^{\mu\nu} &= (\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu)\,(\partial^\mu e_a{}^\nu - \partial^\nu e_a{}^\mu) \\ &= 2\,(\partial_\mu e^a{}_\nu\,\partial^\mu e_a{}^\nu - \partial_\mu e^a{}_\nu\,\partial^\nu e_a{}^\mu) \end{aligned}$$

We can see from this expanded form that the twist invariant is built from two distinct contractions of the first derivatives of the tetrad: the direct contraction $\partial_\mu e^a{}_\nu \partial^\mu e_a{}^\nu$ and the cross contraction $\partial_\mu e^a{}_\nu \partial^\nu e_a{}^\mu$. Each of these supply second-order derivatives, the antisymmetric indices match, and the internal index a is paired with itself leading to a scalar value.

Scale invariant $(\nabla\sigma)^2$**.** Because σ is a true scalar, the contraction $(\nabla_\mu\sigma)$ $(\nabla^\mu\sigma)$ is a true scalar. With the standard measure $\sqrt{-g}$ this yields a weight-+1 Lagrangian density that is diffeomorphism and local-Lorentz invariant:

$$(\nabla_\mu\sigma)(\nabla^\mu\sigma) \;=\; (\nabla_\mu\sigma)^2\,.$$

Second-Order Action We now have all the ingredients we need to write down the action to second-order:

$$S_{(2)} = \int d^4x\,\sqrt{-g}\,\Big[\underbrace{\lambda_S\,S_{\mu\nu}S^{\mu\nu}}_{\text{Shear}} + \underbrace{\frac{1}{4}\,\lambda_T\,T^a{}_{\mu\nu}T_a{}^{\mu\nu}}_{\text{Twist}} + \underbrace{\lambda_C\,(\nabla_\mu\sigma)(\nabla^\mu\sigma)}_{\text{Scale}}\Big]. \tag{5}$$

Note that we have chosen the metric signature $(-+++)$ so that, in flat space, the quadratic action's kinetic terms behave like normal relativistic wave equations if the λ coefficients are positive. As a reminder, for index operations we use,

$$S^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}S_{\rho\sigma}, \quad T_{a\mu\nu} = \eta_{ab}\,T^b{}_{\mu\nu}, \quad T^{a\mu\nu} = g^{\mu\rho}g^{\nu\sigma}T^a{}_{\rho\sigma}, \quad T^\rho{}_{\mu\nu} = e_a{}^\rho\,T^a{}_{\mu\nu}.$$

Finally, as we discussed before, we simplify our bookkeeping by keeping the teleparallel gauge, $\omega^a{}_{b\mu} = 0$, which is why there is no quadratic term built from the spin connection.

4.2 Second-Order Field Equations

We vary the action with respect to shear, twist, and compression to obtain the corresponding field equations. Each of our three invariants is independent of the others, so we proceed to vary each term individually.

Shear Field Equation ($\delta S_{(2),S}$) The shear sector for the action is

$$S_{(2),S} \;=\; \lambda_S \int d^4x\,\sqrt{-g}\;S_{\mu\nu}S^{\mu\nu}, \qquad S_{\mu\nu} \;=\; \nabla_{(\mu}u_{\nu)}.$$

We vary the action with respect to u^ν (holding $g_{\mu\nu}$ fixed in this sector):

$$\delta S_{(2),S} = 2\lambda_S \int d^4x\,\sqrt{-g}\,S^{\mu\nu}\,\nabla_{(\mu}\delta u_{\nu)} = 2\lambda_S \int d^4x\,\sqrt{-g}\,S^{\mu\nu}\,\nabla_\mu \delta u_\nu,$$

where we used the symmetry of $S^{\mu\nu}$. Integrating by parts and discarding the boundary term,

$$\delta S_{(2),S} = -2\lambda_S \int d^4x\,\sqrt{-g}\,\left(\nabla_\mu S^{\mu\nu}\right)\delta u_\nu.$$

Our resulting Euler–Lagrange equation is therefore,

$$\nabla_\mu S^{\mu\nu} \;=\; 0. \tag{6}$$

Equivalently, using $[\nabla_\mu, \nabla^\nu]u^\mu = R^\nu{}_\rho u^\rho$,

$$\nabla_\mu S^{\mu\nu} = 0 \iff \Box u^\nu + \nabla^\nu(\nabla\cdot u) + R^\nu{}_\rho\, u^\rho = 0.$$

In the weak field limit, we have,

$$g_{\mu\nu} \;\to\; \eta_{\mu\nu}, \qquad \nabla_\mu \;=\; \partial_\mu,$$

which gives us,

$$\partial_\mu S^{\mu\nu} \;=\; 0, \qquad S^{\mu\nu} \;=\; \frac{1}{2}\left(\partial^\mu u^\nu + \partial^\nu u^\mu\right).$$

In this limit, Equation (6) becomes,

$$\partial_\mu \partial^\mu u^\nu + \partial^\nu\left(\partial\cdot u\right) \;=\; 0.$$

Choosing Lorenz gauge, $\partial\cdot u = 0$, the shear field equation recovers the d'Alembert equation:

$$\Box\, u^\nu \;=\; 0, \qquad \Box \;\equiv\; \partial_\mu\partial^\mu. \tag{7}$$

Taking this process one step further, to recover the linearized Einstein equations, we can perturb the metric and take the reverse trace:

$$h_{\mu\nu} \ \equiv \ 2S_{\mu\nu} \ = \ \partial_\mu u_\nu + \partial_\nu u_\mu, \qquad \bar{h}_{\mu\nu} \ \equiv \ h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

Taking the divergence of (6) in flat space gives the divergence constraint

$$\partial_\mu h^{\mu\nu} \ = \ 0.$$

The standard harmonic (Lorenz) gauge is $\partial_\mu \bar{h}^{\mu\nu} = 0$; in the case $\partial\cdot u = 0$ we have $h = 2\,\partial\cdot u = 0$, so $\bar{h}_{\mu\nu} = h_{\mu\nu}$ and the two conditions coincide. Under this gauge, acting with $\Box$ on $h_{\mu\nu} = \partial_\mu u_\nu + \partial_\nu u_\mu$ and using (7) yields

$$\Box\,\bar{h}_{\mu\nu} = 0, \tag{8}$$

which matches the vacuum linearized Einstein equation in harmonic gauge.[4] This shows how the shear sector of the action is that it transports massless spin-2 excitations through the structured vacuum.

Twist Field Equation ($\delta S_{(2),T}$) We next proceed to vary the twist sector of the action. Recall that, to simplify our bookkeeping—and without loss of generality—, we work in a flat spin connection. We start by defining,

$$\mathcal{D}_\mu X_a{}^\nu \ \equiv \ \nabla_\mu X_a{}^\nu + \omega_a{}^b{}_\mu X_b{}^\nu, \qquad R^a{}_{b\mu\nu}\,[\omega] \ = \ 0,$$

where ∇_μ is the Levi–Civita derivative on space-time indices, ω is the flat spin connection, and $\mathcal{D}_\mu$ reduces to ∇_μ in the inertial gauge ($\omega = 0$), and to ∂_μ further in Minkowski coordinates where $\Gamma = 0$.

The twist sector of the action is given by,

$$S_{(2),T} \ = \ \frac{1}{4}\,\lambda_T \int d^4x\,\sqrt{-g}\;T^a{}_{\mu\nu}T_a{}^{\mu\nu},$$

which, when we vary with respect to $e^a{}_\nu$ (holding $g_{\mu\nu}$ fixed), we arrive at,

$$\delta T^a{}_{\mu\nu} \ = \ 2\,\mathcal{D}_{[\mu}\delta e^a{}_{\nu]}, \qquad \therefore\ \delta\left(T^a{}_{\mu\nu}T_a{}^{\mu\nu}\right) \ = \ 4\,T_a{}^{\mu\nu}\,\mathcal{D}_\mu\delta e^a{}_\nu,$$

where the factor of 4 comes from the product 2×2 with the definition of $T^a{}_{\mu\nu}$ contributing one factor of 2 and the product rule contributing another.

[4] In Section 5, we show how the quartic action recovers the Einstein–Hilbert action [2, 3].

Inserting this into the action and integrating by parts, we get,

$$\delta S_{(2),T} = \lambda_T \int d^4x \, \sqrt{-g} \, T_a{}^{\mu\nu} \mathcal{D}_\mu \delta e^a{}_\nu$$
$$= -\lambda_T \int d^4x \, \sqrt{-g} \, \left(\mathcal{D}_\mu T_a{}^{\mu\nu} \right) \delta e^a{}_\nu,$$

where the boundary term vanishes to ensure that only bulk terms contribute to the field equation. Our resulting Euler–Lagrange equation becomes,

$$\mathcal{D}_\mu T_a{}^{\mu\nu} \; = \; 0. \tag{9}$$

with $\mathcal{D}_\mu T_a{}^{\mu\nu} = \nabla_\mu T_a{}^{\mu\nu} + \omega_a{}^b{}_\mu \, T_b{}^{\mu\nu}$.

As before, we can take the weak field limit, where,

$$g_{\mu\nu} \; \to \; \eta_{\mu\nu}, \qquad \omega_a{}^{b\mu} \; \to \; 0 \quad \text{(inertial gauge)},$$

and,

$$\mathcal{D}_\mu \; = \; \partial_\mu, \qquad T^a{}_{\mu\nu} \; = \; 2\,\partial_{[\mu} e^a{}_{\nu]}, \qquad \partial_\mu T_a{}^{\mu\nu} \; = \; 0. \tag{10}$$

Pick any fixed internal unit vector[5] n_a and define the abelian projection

$$A_\mu[n] \; \equiv \; n_a \, e^a{}_\mu, \qquad F_{\mu\nu}[n] \; \equiv \; n_a \, T^a{}_{\mu\nu}.$$

With flat spin connection ($R[\omega] = 0$) and a covariantly constant internal vector n_a (that is, $\mathcal{D}_\mu n_a = 0$), the twist equations project to

$$\mathcal{D}_\mu F^{\mu\nu}[n] = 0, \qquad \mathcal{D}_{[\lambda} F_{\mu\nu]}[n] = 0.$$

In the inertial gauge $\omega \to 0$ this reduces to $\partial_\mu F^{\mu\nu}[n] = 0$ and $\partial_{[\lambda} F_{\mu\nu]}[n] = 0$. For notational simplicity we take $n_a = \delta_a^{\hat{0}}$ below, so $A_\mu \equiv A_\mu[n]$ and $F_{\mu\nu} \equiv F_{\mu\nu}[n]$. In this limit, Equation (10) recovers the vacuum Maxwell equations [4] in our working limit where $\omega = 0$,

$$\partial_\mu F^{\mu\nu}[n] = 0, \qquad \partial_{[\lambda} F_{\mu\nu]}[n] = 0, \quad \text{(for any constant } n_a\text{)}.$$

This shows that the twist sector transports massless spin-1 excitations through the structured vacuum.

[5] The internal index a transforms in the vector representation of the local Lorentz group $\mathrm{SO}(1,3)$. Picking any covariantly constant unit vector n_a defines an abelian U(1) subsector via $A_\mu[n] = n_a e^a{}_\mu$ and $F_{\mu\nu}[n] = n_a T^a{}_{\mu\nu}$. This is *not* a Yang–Mills SU(2): there are no nonabelian self-interactions here. Our choice $n_a = \delta_a^{\hat{0}}$ is merely a convenient projection; any such n_a works.

Scale Field Equation ($\delta S_{(2),C}$) Finally, we vary the scale sector of the action,

$$S_{(2),C} \;=\; \lambda_C \int d^4x \, \sqrt{-g} \; (\nabla_\mu \sigma)\,(\nabla^\mu \sigma)\,,$$

like so,

$$\delta\left[(\nabla\sigma)^2\right] \;=\; 2\nabla_\mu\sigma\nabla^\mu\delta\sigma.$$

Inserting this back into the action, and integrating by parts, we get,

$$\begin{aligned}\delta S_{(2),C} &= 2\lambda_C \int d^4x \, \sqrt{-g} \; \nabla_\mu\sigma\nabla^\mu\delta\sigma \\ &= -2\lambda_C \int d^4x \, \sqrt{-g} \; (\nabla_\mu\nabla^\mu\sigma)\,\delta\sigma.\end{aligned}$$

once again taking the surface term to be 0 to ensure that only bulk terms contribute to the field equation. The Euler–Lagrange equation for the scale term is then,

$$\nabla_\mu\nabla^\mu\sigma \;=\; 0, \tag{11}$$

which is the covariant massless Klein–Gordon equation for a scalar field σ.

In the weak field limit, we once again have,

$$g_{\mu\nu} \;\to\; \eta_{\mu\nu}, \qquad \nabla_\mu \;=\; \partial_\mu,$$

which results in the flat-space equations,

$$\Box\sigma \;=\; 0, \qquad \Box \;\equiv\; \partial_\mu\partial^\mu.$$

We next seek to relate the scale field equation of motion to its Bergmann–Wagoner [5, 6, 7] form. For this mapping we use the Einstein–Hilbert [2, 3] term together with the scale kinetic term:

$$S_{(2),SC} \;\equiv\; \lambda_S \int d^4x \sqrt{-g}\; R(g) \;+\; \lambda_C \int d^4x \sqrt{-g}\; (\nabla\sigma)^2.$$

We factor out the conformal component by writing the full tetrad, $e^a{}_\mu$, as a product of an overall scale $\sigma^{1/3}$ times a unit-determinant tetrad:

$$e^a{}_\mu \;=\; \sigma^{1/3}\,\tilde{e}^a{}_\mu, \qquad \det\left(\tilde{e}^a{}_\mu\right) = 1.$$

Here $\sigma > 0$ denotes a scale factor, not the dilaton defined earlier as $\sigma = \ln(|e|/|e|_0)$; they are related by $|e| = \sigma^{4/3} \Rightarrow \sigma = \big(|e|/|e|_0\big)^{3/4}$. The metric then decomposes as,

$$g_{\mu\nu} \;=\; \sigma^{2/3}\,\tilde{g}_{\mu\nu}, \qquad \sqrt{-g} \;=\; \sigma^{4/3}\,\sqrt{-\tilde{g}},$$

where the tilde quantities carry no overall scale and all dilations now live in the single scalar field $\sigma(x)$.

Using the Weyl-rescaling identity for the Ricci scalar ($\Omega = \sigma^{1/3}$),

$$R(g) \;=\; \Omega^{-2}\big[\tilde{R} - 6\,\tilde{\nabla}^2 \ln\Omega - 6(\tilde{\nabla}\ln\Omega)^2\big],$$

and keeping the Laplacian as a boundary term, the gravitational term becomes

$$\begin{aligned}
\lambda_S \int d^4x\,\sqrt{-g}\;R(g) &= \lambda_S \int d^4x\,\sigma^{4/3}\sqrt{-\tilde{g}}\;\times\;\sigma^{-2/3}\Big[\tilde{R} + 6\big(\tilde{\nabla}\ln\sigma^{1/3}\big)^2\Big]\\
&= \lambda_S \int d^4x\,\sigma^{2/3}\sqrt{-\tilde{g}}\;\Big[\tilde{R} + \frac{2}{3}\,\sigma^{-2}\big(\tilde{\nabla}\sigma\big)^2\Big]\\
&= \lambda_S \int d^4x\,\sqrt{-\tilde{g}}\;\Big[\sigma^{2/3}\,\tilde{R} + \frac{2}{3}\,\sigma^{-4/3}\big(\tilde{\nabla}\sigma\big)^2\Big].
\end{aligned}$$

We next factor out the conformal component from the scale term:

$$\lambda_C \int d^4x\,\sqrt{-g}\;\left(\nabla_\mu\sigma\right)\left(\nabla^\mu\sigma\right) = \;\lambda_C \int d^4x\,\sqrt{-\tilde{g}}\;\sigma^{2/3}\left(\tilde{\nabla}_\mu\sigma\right)\left(\tilde{\nabla}^\mu\sigma\right).$$

In order to have a canonical Bergmann–Wagoner [5, 6, 7] structure, we rescale the scalar,

$$\phi \;\coloneqq\; \sigma^{2/3}, \qquad \tilde{\nabla}_\mu\sigma \;=\; \frac{3}{2}\,\phi^{1/2}\,\tilde{\nabla}_\mu\,\phi, \qquad \left(\tilde{\nabla}\sigma\right)^2 =\; \frac{9}{4}\,\phi\left(\tilde{\nabla}\phi\right)^2.$$

The total action then becomes,

$$S_{(2),SC} \;=\; \int d^4x\,\sqrt{-\tilde{g}}\,\left\{\lambda_S\Big[\phi\,\tilde{R} + \frac{3}{2}\,\phi^{-1}\big(\tilde{\nabla}\phi\big)^2\Big] \;+\; \lambda_C\Big[\frac{9}{4}\,\phi^2\big(\tilde{\nabla}\phi\big)^2\Big]\right\}.$$

We observe that the Jordan-frame Brans–Dicke action is,

$$S_{(2),\mathrm{BD}} \;=\; \frac{1}{16\pi}\,\int d^4x\,\sqrt{-\tilde{g}}\,\left[\phi\,\tilde{R} - \frac{\omega\left(\phi\right)}{\phi}\left(\tilde{\nabla}\phi\right)^2\right].$$

Comparing the two actions, we identify λ_S and the field-dependent $\omega(\phi)$ as,

$$\lambda_S \;=\; \frac{1}{16\pi}, \qquad \omega\,(\phi) \;=\; -\frac{3}{2} - 36\pi\,\lambda_C\,\phi^3.$$

λ_C therefore sets the scalar sector. When $\lambda_C = 0$ we obtain Brans–Dicke with $\omega_{\rm BD} = -\frac{3}{2}$; for large λ_C the scalar is driven to (nearly) constant, recovering Einstein gravity with an effective Newton constant rescaled by that constant ϕ.

This completes our derivation of the second-order field equations for the structured vacuum. We next turn to the quartic action, which will allow us to recover the Einstein–Hilbert [2, 3] action of general relativity. In addition, we will show how the quartic action supports new physics, such as vacuum birefringence, light-by-light scattering [21], and spin–quadrupole gravitational radiation.

5 Quartic Vacuum Dynamics

We approach the design of our quartic vacuum dynamics using an approach similar to our quadratic dynamics. We start with our three first-derivative tensor building blocks from Section 4.1 (with $T^a{}_{\mu\nu} \equiv 2\,\mathcal{D}_{[\mu}e^a{}_{\nu]}$; in the inertial gauge $\omega^{ab}{}_\mu = 0$ we have $\mathcal{D}_\mu e^a{}_\nu = \nabla_\mu e^a{}_\nu$ and hence $T^a{}_{\mu\nu} = 2\nabla_{[\mu}e^a{}_{\nu]}$):

$$S_{\mu\nu} \equiv \nabla_{(\mu}u_{\nu)}, \qquad T^a{}_{\mu\nu} \equiv 2\nabla_{[\mu}e^a{}_{\nu]}, \qquad \nabla_\mu\sigma.$$

We then seek a local Lagrangian that is a quartic polynomial in the base first derivatives (and quadratic in these three tensors).

5.1 Quartic Action

To build an effective quartic action we want one—and only one—term for each truly different way any fields can interact. So, in addition to the necessary and sufficient criteria of locality, covariance, and scalarity, we imposed for the quadratic action in Section 4.1, we impose an additional set of four criteria that additionally apply at the fourth-order:

1. **Integral Independence.** We discard as redundant any terms that differ only by a total divergence $\partial_\alpha\,(\dots)$. Because the surface term vanishes for finite-energy field configurations (such as the ones we assume), the two terms produce identical equations of motion.

2. **Algebraic Uniqueness.** We avoid index contractions that cause a linear combination to collapse to zero (for example: symmetry, anti-symmetry, trace, Schouten). Since these terms are the same tensor written in two ways, keeping both just renames a coefficient, so we leave them out.

3. **Field Redefinition.** We remove from the action any terms that equal another term plus a value proportional to the quadratic-order equations of motion. We do this because we can use the quadratic equations of motion to move that value into a redefinition of the fields or parameters.

4. **Symmetry Projection.** We exclude any terms that flip sign under parity or time-reversal. This is because if the vacuum and experimental data respect parity and time-reversal, then the odd term cannot appear (or, if it does, its coefficient must be zero).

Restricting to expressions quartic in first derivatives, the independent scalar monomials are the quadratic products of the three quadratic scalars,

$$S^2 \equiv S_{\mu\nu}S^{\mu\nu}, \qquad T^2 \equiv T^a{}_{\mu\nu}T_a{}^{\mu\nu}, \quad \text{and} \quad (\nabla\sigma)^2 \equiv \nabla_\mu\sigma\,\nabla^\mu\sigma.$$

Thus the six basic quartic monomials are

$$(S^2)^2, \qquad (T^2)^2, \qquad \big((\nabla\sigma)^2\big)^2, \qquad S^2T^2, \qquad S^2(\nabla\sigma)^2, \qquad T^2(\nabla\sigma)^2.$$

Applying our fourth-order necessary and sufficient criteria, we are left with only the following invariants.[6]

Shear Invariant $S_{\mu\nu}S^{\nu\rho}S_{\rho\sigma}S^{\sigma\mu} - S_{\mu\nu}S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma}$. We contract the quadratic-order shear tensor with itself to get the quartic-order shear invariant. Any total divergence of an S^3 product reduces to a cubic-order term $+\partial S$, which gets eliminated after integration by parts, so our integral independence criterion is satisfied.

In terms of the algebraic uniqueness criterion, there are two possible S^4 traces and they are linearly independent, so we choose the Weyl-type difference in order to remove any double counting under index symmetry. Since $\nabla_\mu S^{\mu\nu} = 0$ cannot turn S^4 into another scalar, we cannot use field redefinition to eliminate this term, so we keep it.

Finally, the shear invariant is even parity and time-reversal; it meets all of our quartic-order necessary and sufficient criteria.

[6] For a listing of the candidates that do not satisfy these criteria, see Appendix A.

Twist Invariants $(T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu})^2 + (T\cdot\tilde{T})^2$**.** We contract the quadratic-order twist tensor with itself to get the quartic-order twist invariant. Any divergence of a T^3 term vanishes by antisymmetry, or integration by parts does not produce any duplicated terms.

Any alternative arrangement of the T^4 indices either collapses to the twist invariant or zero via antisymmetry, so the twist invariant is algebraically unique. The twist invariant cannot be eliminated by field redefinition because $\partial_\mu T_{\alpha}{}^{\mu\nu} = 0$ never appears inside T^2.

Finally, the twist invariant is even under parity and time reversal. A second parity-even invariant in four dimensions is $\left(T \cdot \tilde{T}\right)^2$ with $\tilde{T}^{a}{}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}T^{a\,\rho\sigma}$. While at the quadratic order, we excluded the linear pseudoscalar $T\cdot\tilde{T}$ by symmetry projection; its square is allowed in the quartic order and we keep it.

Scale Invariant $(\nabla_\mu\sigma\nabla^\mu\sigma)^2$**.** Similarly, we contract the quadratic-order scale tensor with itself to get the quartic-order scale invariant. We discard the candidate $(\Box\sigma)^2$ because it differs from our chosen scale invariant only by a total divergence after integration by parts, making it not integral-independent.

There is only one way to contract four gradients of a single scalar, so the scale invariant satisfies algebraic uniqueness. At quadratic order the scale equation of motion is $\Box\sigma = 0$, since our proposed quartic-order scale invariant does not contain $\Box\sigma$, it passes our field redefinition criteria.

And since the scale invariant is a scalar, it is parity-even automatically, satisfying symmetry projection.

Helixity Invariant $S^{\mu\nu}S_{\mu\nu}T^{\alpha}{}_{\rho\sigma}T_{\alpha}{}^{\rho\sigma}$**.** The helixity invariant is a product of the shear and twist invariants. The form we keep satisfied integral independence because it does not reduce to an S^2 or T^2 term after integration by parts. In terms of algebraic uniqueness, the helixity invariant is the only independent scalar with two S and two T factors after index symmetry.

Neither $\partial_\mu S^{\mu\nu}$ nor $\partial_\mu T^{\mu\nu}$ are proportional to the quadratic-order equations of motion, so the helixity invariant passes the field redefinition criterion.

Finally, the helixity invariant satisfies symmetry projection as it is even parity and time-reversal.

Strain Invariant $S_{\mu\nu}S^{\mu\nu}\nabla_\rho\sigma\nabla^\rho\sigma$. The strain invariant is a product of the shear and scale invariants. The form we keep satisfied integral independence because its total derivative does not produce a term like $S^{\mu\nu}\nabla_\mu S_{\nu\rho}\nabla^\rho\sigma$ that contains an equation of motion.

In terms of algebraic uniqueness, the strain invariant is the only non-zero symmetric contraction of two S and two $\nabla\sigma$ factors. The strain invariant is safe from field redefinition because neither quadratic equation of motion for shear or scale can be factored out of it.

Finally, the strain invariant satisfies symmetry projection as it is parity-even.

Spirality Invariant $T^\alpha{}_{\rho\sigma}T_\alpha{}^{\rho\sigma}\nabla_\rho\sigma\nabla^\rho\sigma$. The spirality invariant is a product of the twist and scale invariants. Similar logic to the strain invariant applies here: The form we keep satisfied integral independence because its total derivative does not produce a term like $T^\alpha{}_{\rho\sigma}\nabla_\alpha T^{\rho\sigma}_\nu\nabla^\nu\sigma$ that contains an equation of motion.

In addition, the spirality invariant is the only independent symmetric scalar with T^2 and $(\partial\sigma)^2$ factors, so it satisfies algebraic uniqueness. The spirality invariant is safe from field redefinition because it does not contain any on-shell factors of the quadratic equations of motion for twist or scale.

Finally, the spirality invariant satisfies symmetry projection as it is parity-even.

This concludes our search for quartic-order invariants. We next move on to write the full quartic action, which will allow us to recover the Einstein–Hilbert [2, 3] action of general relativity. In addition, we will show how the quartic action supports new physics, such as vacuum birefringence, light-by-light scattering [21], and spin–quadrupole gravitational radiation.

Fourth-Order Action We can now write the full quartic action:

$$
\begin{aligned}
S_{(4)} \;=\; \int d^4x\,\sqrt{-g}\,\Bigg\{ & \underbrace{\alpha_S\Big(S_{\mu\nu}S^{\nu\rho}S_{\rho\sigma}S^{\sigma\mu} - S_{\mu\nu}S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma}\Big)}_{\text{Shear}} + \\
& \underbrace{\frac{1}{4}\alpha_T\left[\Big(T^a{}_{\mu\nu}T_a{}^{\mu\nu}\Big)^2 + \Big(T^a{}_{\mu\nu}\tilde{T}_a{}^{\mu\nu}\Big)^2\right]}_{\text{Twist}} + \\
& \underbrace{\alpha_C\big(\nabla_\mu\sigma\,\nabla^\mu\sigma\big)^2}_{\text{Scale}} + \underbrace{\alpha_H\,S^{\mu\nu}S_{\mu\nu}\,T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}}_{\text{Helixity}} + \\
& \underbrace{\alpha_R\,S_{\mu\nu}S^{\mu\nu}\,\nabla_\rho\sigma\,\nabla^\rho\sigma}_{\text{Strain}} + \underbrace{\alpha_P\,T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\,\nabla_\rho\sigma\,\nabla^\rho\sigma}_{\text{Spirality}} \Bigg\}.
\end{aligned}
\tag{12}
$$

5.2 Fourth-Order Field Equations

As before, we vary the action with respect to shear, twist, scale, then, additionally, the new invariants helixity, strain, and spirality to obtain the corresponding field equations. Each of our six invariants is independent of the others, so we proceed to vary each term individually.

5.2.1 Shear as a Gravitational Field

We begin by varying the quartic action with respect to the shear tensor, holding $g_{\mu\nu}$ fixed in this sector,

$$
S_{\mu\nu} \;=\; \nabla_{(\mu}u_{\nu)}, \qquad \delta S_{\mu\nu} \;=\; \nabla_{(\mu}\delta u_{\nu)}
$$

Terms in the action that do not contain the shear tensor do not contribute to the shear field equation.

We write the shear-dependent part of the action,

$$
\begin{aligned}
S_{(4),S} \;=\; \int d^4x\,\sqrt{-g}\,\Big[& \alpha_S\left(S_{\mu\nu}S^{\nu\rho}S_{\rho\sigma}S^{\sigma\mu} - S_{\mu\nu}S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma}\right) \\
& + \alpha_H S_{\mu\nu}S^{\mu\nu}T^a{}_{\rho\sigma}T_a{}^{\rho\sigma} + \alpha_R S_{\mu\nu}S^{\mu\nu}\nabla_\rho\sigma\nabla^\rho\sigma\Big].
\end{aligned}
$$

As we can see, in addition to pure shear, the shear-dependent part of the action also contains helixity and strain.

For the pure shear term, we have,

$$\delta(S_{\mu\nu}S^{\nu\rho}S_{\rho\sigma}S^{\sigma\mu} - S_{\mu\nu}S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma}) \;=\; 4\,(S^{\mu\rho}S_{\rho\sigma}S^{\sigma\nu} - S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma})\,\delta S_{\mu\nu}.$$

and for the mixed terms we have,

$$\delta\left(S_{\mu\nu}S^{\mu\nu}T^{a}{}_{\rho\sigma}T_{a}{}^{\rho\sigma}\right) = 2T^{a}{}_{\rho\sigma}T_{a}{}^{\rho\sigma}S^{\alpha\beta}\delta S_{\alpha\beta},$$
$$\delta\left(S_{\mu\nu}S^{\mu\nu}\nabla_{\rho}\sigma\nabla^{\rho}\sigma\right) = 2\nabla_{\rho}\sigma\nabla^{\rho}\sigma S^{\alpha\beta}\delta S_{\alpha\beta}.$$

Let

$$C^{\mu\nu} \;=\; 4\alpha_S(S^{\mu\rho}S_{\rho\sigma}S^{\sigma\nu} - S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma}) + 2\alpha_H\,T^{a}{}_{\rho\sigma}T_{a}{}^{\rho\sigma}S^{\mu\nu} + 2\alpha_R\,(\nabla\sigma)^2 S^{\mu\nu}.$$

Using the symmetry of $C^{\mu\nu}$, we have $C^{\mu\nu}\nabla_{(\mu}\delta u_{\nu)} = C^{\mu\nu}\nabla_{\mu}\delta u_{\nu}$. Therefore

$$\delta S_{(4),S} \;=\; \int d^4x\,\sqrt{-g}\;C^{\mu\nu}\,\nabla_{\mu}\delta u_{\nu},$$

Integrating by parts (and discarding the boundary term) yields

$$\delta S_{(4),S} \;=\; -\int d^4x\,\sqrt{-g}\,\left(\nabla_{\mu}C^{\mu\nu}\right)\delta u_{\nu},$$

so the Euler–Lagrange equation is $\nabla_{\mu}C^{\mu\nu} = 0$.

Writing the full Euler–Lagrange equation for the shear field, we have,

$$\nabla_{\mu}\Big[4\alpha_S\left(S^{\mu\rho}S_{\rho\sigma}S^{\sigma\nu} - S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma}\right) + 2\alpha_H T^{a}{}_{\rho\sigma}T_{a}{}^{\rho\sigma}S^{\mu\nu} + 2\alpha_R\nabla_{\rho}\sigma\nabla^{\rho}\sigma S^{\mu\nu}\Big] \;=\; 0. \tag{13}$$

We next set out to relate the pure shear part of the shear field Euler–Lagrange equation to Einstein–Hilbert [2, 3]. We start by isolating the pure shear part,

$$\nabla_{\mu}\left[4\alpha_S\left(S^{\mu\rho}S_{\rho\sigma}S^{\sigma\nu} - S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma}\right)\right] \;=\; 0.$$

We then relate $S_{\mu\nu}$ to the metric perturbation $h_{\mu\nu}$:

$$S_{\mu\nu} \;\equiv\; \frac{1}{2}h_{\mu\nu}, \qquad S_{\rho\sigma}S^{\rho\sigma} \;=\; \frac{1}{4}h_{\rho\sigma}h^{\rho\sigma}, \qquad S^{\nu\rho}S_{\rho\sigma}S^{\sigma\mu} \;=\; \frac{1}{8}h^{\nu\rho}h_{\rho\sigma}h^{\sigma\mu},$$

to get,

$$4\alpha_S\left(\frac{1}{8}\,h^{\mu\rho}h_{\rho\sigma}h^{\sigma\nu} - \frac{1}{8}\,h^{\mu\nu}\,h_{\rho\sigma}h^{\rho\sigma}\right) \;=\; \alpha_S\left(\frac{1}{2}\,h^{\mu\rho}h_{\rho\sigma}h^{\sigma\nu} - \frac{1}{2}\,h^{\mu\nu}h_{\rho\sigma}h^{\rho\sigma}\right). \tag{14}$$

We next write the Einstein–Hilbert action in terms of $h_{\mu\nu}$. For the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the curvature scalar expands to,

$$\sqrt{-g}\;R \;=\; -\frac{1}{4}h_{\mu\nu}\Box\bar{h}^{\mu\nu} + \frac{1}{2}h^{\mu\rho}h_{\rho\sigma}h^{\sigma\nu} - \frac{1}{2}h^{\mu\nu}h_{\rho\sigma}h^{\rho\sigma} + \mathcal{O}\big(h^4\big),$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Notice how the pure shear field equation, when related to the metric perturbation in Equation (14), is proportional to cubic part of the Einstein–Hilbert action.

We next vary the Einstein–Hilbert action with respect to $h_{\mu\nu}$. Since the action is,

$$S_{EH} \;=\; \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\;R,$$

when we take the variation, we have,

$$\frac{\delta S_{EH}}{\delta h_{\mu\nu}} \;=\; -\frac{1}{32\pi G}\,\Box\,\bar{h}^{\mu\nu} + \frac{1}{16\pi G}\left(\frac{1}{2}h^{\mu\rho}h_{\rho\sigma}h^{\sigma\nu} - \frac{1}{2}h^{\mu\nu}h_{\rho\sigma}h^{\rho\sigma}\right). + \mathcal{O}\big(h^3\big).$$

Recall the quadratic-order shear field equation that reproduced the linearized Einstein equation[2, 3] in Equations (6) and (8):

$$\nabla_\mu S^{\mu\nu} \;=\; 0 \qquad \Longleftrightarrow \qquad \Box\,\bar{h}_{\mu\nu} \;=\; 0.$$

Setting $\lambda_S = \frac{1}{32\pi G}$ and $\alpha_S = \frac{1}{16\pi G}$, we see that the pure shear part of the quadratic- and quartic-order field equations of the structured vacuum combine to give us the Einstein–Hilbert action:

$$\begin{aligned}\frac{\delta S_S}{\delta u^\nu} \;&=\; \delta S_{(2),S} + \delta S_{(4),S} + \mathcal{O}\big(h^3\big)\\ &=\; -\lambda_S\left(\nabla_\mu S^{\mu\nu}\right) + \alpha_S\left(\frac{1}{2}h^{\mu\rho}h_{\rho\sigma}h^{\sigma\nu} - \frac{1}{2}h_{\mu\nu}h_{\rho\sigma}h^{\rho\sigma}\right)\\ &=\; -\frac{1}{32\pi G}\left(\nabla_\mu S^{\mu\nu}\right) + \frac{1}{16\pi G}\left(\frac{1}{2}h^{\mu\rho}h_{\rho\sigma}h^{\sigma\nu} - \frac{1}{2}h_{\mu\nu}h_{\rho\sigma}h^{\rho\sigma}\right).\end{aligned}$$

With those choices for λ_S and α_S, the total variation of $S_{(2),S} + S_{(4),S}$ equals δS_{EH}, and we have,

$$\frac{\delta S_S}{\delta u^\nu} \;=\; \frac{\delta S_{EH}}{\delta h_{\mu\nu}} \;=\; 0 \quad\Longleftrightarrow\quad G_{\mu\nu} \;=\; 0,$$

where $G_{\mu\nu}$ is the Einstein vacuum equation. Thus, we have shown that the Einstein–Hilbert theory arises directly as the pure-shear limit of the structured vacuum, establishing spin-2 gravitation as the shear sector of the structured vacuum.

5.2.2 Twist as a Gauge Field

We now revisit the twist-dependent sector of the quartic action. Introducing the Hodge dual $\tilde{T}^a{}_{\mu\nu} \equiv \frac{1}{2} E_{\mu\nu}{}^{\rho\sigma} T^a{}_{\rho\sigma}$ with $E_{\mu\nu\rho\sigma} = \sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$, we include the second parity-even quartic invariant:

$$S_{(4),T} = \int d^4x\,\sqrt{-g}\,\Bigg\{\frac{1}{4}\,\alpha_T\left[\left(T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\right)^2 + \left(T^a{}_{\rho\sigma}\tilde{T}_a{}^{\rho\sigma}\right)^2\right] \\ +\ \alpha_H\, S^{\mu\nu}S_{\mu\nu}\, T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\ +\ \alpha_P\, T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\,\nabla_\rho\sigma\,\nabla^\rho\sigma\Bigg\},$$

where $T^a{}_{\mu\nu} \equiv 2\,\nabla_{[\mu}e^a{}_{\nu]}$ and $S^{\mu\nu}$ and σ are background fields that do not vary in the present twist variation.

Define the scalar invariant $Q \equiv T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}$. Define also the pseudoscalar $B \equiv T^a{}_{\rho\sigma}\tilde{T}_a{}^{\rho\sigma}$. Using $\delta T^a{}_{\mu\nu} = 2\,\nabla_{[\mu}\delta e^a{}_{\nu]}$, we have

$$\delta Q = 2\,T_a{}^{\rho\sigma}\,\delta T^a{}_{\rho\sigma} = 4\,T_a{}^{\rho\sigma}\,\nabla_\rho \delta e^a{}_\sigma,$$

where the last equality follows from antisymmetry in $\rho\sigma$. Therefore,

$$\begin{aligned} \delta\left[\frac{1}{4}\alpha_T Q^2\right] &= \frac{1}{2}\alpha_T\, Q\,\delta Q = 2\,\alpha_T\, Q\, T_a{}^{\mu\nu}\,\nabla_\mu\delta e^a{}_\nu, \\ \delta\left[\frac{1}{4}\alpha_T B^2\right] &= \frac{1}{2}\alpha_T\, B\,\delta B = 2\,\alpha_T\, B\,\tilde{T}_a{}^{\mu\nu}\,\nabla_\mu\delta e^a{}_\nu, \\ \delta\big[\alpha_H S^2\, T\cdot T\big] &= 4\,\alpha_H\, S^{\rho\sigma}S_{\rho\sigma}\, T_a{}^{\mu\nu}\,\nabla_\mu\delta e^a{}_\nu, \\ \delta\big[\alpha_P(\nabla\sigma)^2\, T\cdot T\big] &= 4\,\alpha_P\,(\nabla\sigma)^2\, T_a{}^{\mu\nu}\,\nabla_\mu\delta e^a{}_\nu, \end{aligned} \tag{15}$$

with $(\nabla\sigma)^2 \equiv \nabla_\rho\sigma\,\nabla^\rho\sigma$.

Inserting (15) into $\delta S_{(4),T}$ and integrating by parts (discarding boundary terms) gives

$$\delta S_{(4),T} = -\int d^4x\,\sqrt{-g}\,\nabla_\mu\Big[2\alpha_T\left(Q\,T_a{}^{\mu\nu} + B\,\tilde{T}_a{}^{\mu\nu}\right) + 4\alpha_H\,S^2\,T_a{}^{\mu\nu} + 4\alpha_P\,(\nabla\sigma)^2\,T_a{}^{\mu\nu}\Big]\,\delta e^a{}_\nu.$$

Since $\delta e^a{}_\nu$ is arbitrary, we obtain

$$\nabla_\mu\Big[2\alpha_T\left(Q\,T_a{}^{\mu\nu} + B\,\tilde{T}_a{}^{\mu\nu}\right) + 4\alpha_H\,S^{\rho\sigma}S_{\rho\sigma}\,T_a{}^{\mu\nu} + 4\alpha_P\,(\nabla\sigma)^2\,T_a{}^{\mu\nu}\Big] \;=\; 0. \tag{16}$$

To expose the gauge structure, choose a fixed internal time-like label $\alpha = \hat{0}$ and define

$$A_\mu \;\equiv\; e^{\hat{0}}{}_\mu, \qquad F_{\mu\nu} \;\equiv\; T^{\hat{0}}{}_{\mu\nu}.$$

Recalling Equation (4),

$$T^a{}_{\mu\nu} = 2\,\nabla_{[\mu}e^a{}_{\nu]} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu,$$

we have

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \nabla_{[\lambda}F_{\mu\nu]} \;=\; 0 \quad \text{(Bianchi identity)}.$$

$$\tilde{F}_{\mu\nu} \;\equiv\; \tfrac{1}{2}\,E_{\mu\nu}{}^{\rho\sigma}\,F_{\rho\sigma}, \qquad E_{\mu\nu\rho\sigma} = \sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}.$$

The $\alpha = \hat{0}$ component of (16) becomes a non-linear gauge equation with a field-dependent constitutive factor,

$$\begin{aligned} &\nabla_\mu\Big[\Lambda\,F^{\mu\nu} + \tilde{\Lambda}\,\tilde{F}^{\mu\nu}\Big] \;=\; 0, \\ &\Lambda \;\equiv\; 2\alpha_T\,Q + 4\alpha_H\,S^{\rho\sigma}S_{\rho\sigma} + 4\alpha_P\,(\nabla\sigma)^2, \quad \tilde{\Lambda} \;\equiv\; 2\alpha_T\,B, \end{aligned} \tag{17}$$

where $Q = T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}$ is the full twist invariant (its variation with respect to $e^{\hat{0}}{}_\mu$ arises only through $T^{\hat{0}}{}_{\rho\sigma} = F_{\rho\sigma}$).

In the pure-twist limit ($\alpha_H = \alpha_P = 0$), the gauge equation becomes

$$\nabla_\mu\,2\alpha_T\left[Q\,F^{\mu\nu} + B\,\tilde{F}^{\mu\nu}\right] = 0.$$

a cubic, non-linear electrodynamics driven by the torsional scalar $Q = T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}$, which rescales the usual Maxwell term $F^{\mu\nu}$, and by the torsional pseudoscalar $B = T^a{}_{\rho\sigma}\tilde{T}_a{}^{\rho\sigma}$, which multiplies the dual $\tilde{F}^{\mu\nu}$ and induces an axion-like magneto-electric mixing [12, 13, 14, 15] (the torsional analogue of $E{\cdot}B$). In parity-even backgrounds $B = 0$ and the $\tilde{F}$ piece vanishes. When $\Lambda = 2\alpha_T Q$ and $\tilde{\Lambda} = 2\alpha_B B$ are effectively constant, such as when $S^{\rho\sigma}S_{\rho\sigma}$ and $(\nabla\sigma)^2$ are constant and Q, B are slowly varying or replaced by background values, the Bianchi identity $\nabla_\mu \tilde{F}^{\mu\nu} = 0$ reduces (17) to

$$\Lambda\, \nabla_\mu F^{\mu\nu} = 0,$$

which—after an overall rescaling—is the vacuum Maxwell equation [4]

$$\nabla_\mu F^{\mu\nu} = 0.$$

5.2.3 Scale as a Dilaton Field

We next turn our attention to the scale field, σ. As a reminder, the scale-dependent portion of the quartic action is,

$$\begin{aligned} S_{(4),C} \;=\; \int d^4x\, \sqrt{-g}\, \Big[& \alpha_C \left(\nabla_\mu\sigma\nabla^\mu\sigma\right)^2 \\ & + \alpha_R S_{\mu\nu}S^{\mu\nu}\nabla_\rho\sigma\nabla^\rho\sigma \\ & + \alpha_P T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\nabla_\rho\sigma\nabla^\rho\sigma \Big], \end{aligned}$$

and it consists of a pure scale term, a strain term, and a spirality term.

For the pure scale term,

$$\begin{aligned} X \;&\equiv\; (\nabla\sigma)^2 = \nabla_\mu\sigma\, \nabla^\mu\sigma, \\ \delta X \;&=\; 2\, \nabla_\mu\sigma\, \nabla^\mu\delta\sigma, \\ \delta(X^2) \;&=\; 4(\nabla\sigma)^2\, \nabla_\mu\sigma\, \nabla^\mu\delta\sigma. \end{aligned}$$

For the strain and spirality terms,

$$\begin{aligned} \delta\big[S_{\mu\nu}S^{\mu\nu}\, (\nabla\sigma)^2\big] \;&=\; 2\, S_{\mu\nu}S^{\mu\nu}\, \nabla_\rho\sigma\, \nabla^\rho\delta\sigma, \\ \delta\big[T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\, (\nabla\sigma)^2\big] \;&=\; 2\, T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\, \nabla_\rho\sigma\, \nabla^\rho\delta\sigma. \end{aligned}$$

Inserting these into the action and integrating by parts (discarding the boundary term), we get

$$\delta S_{(4),C} = -\int d^4x\,\sqrt{-g}\;\nabla_\mu\Big(4\alpha_C(\nabla\sigma)^2\,\nabla^\mu\sigma + 2\alpha_R S_{\rho\sigma}S^{\rho\sigma}\,\nabla^\mu\sigma + 2\alpha_P T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\,\nabla^\mu\sigma\Big)\,\delta\sigma.$$

And the resulting Euler–Lagrange equation for the scale field is

$$\nabla_\mu\Big(4\alpha_C(\nabla\sigma)^2\,\nabla^\mu\sigma + 2\alpha_R S_{\rho\sigma}S^{\rho\sigma}\,\nabla^\mu\sigma + 2\alpha_P T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\,\nabla^\mu\sigma\Big) = 0.$$

This concludes our analysis of the structured vacuum quartic action, Equation (12). We have shown that the structured vacuum naturally leads to the emergence of relativistic gravity from its pure shear interactions, electromagnetism from its pure twist interactions, and that the scale field σ behaves as a dilaton field. We next turn to some of the falsifiable predictions that emerge from the quartic action.

5.3 Falsifiable Predictions

At quadratic order the structured vacuum behaves linearly: shear (spin-2), twist (spin-1), and scale (scalar) excitations propagate without interacting with each other. At the quartic order, however, the local “stiffness” of the medium depends—very weakly—on the energy carried by the fields themselves: Strong fields slightly reshape the medium through which other fields travel. That single idea underlies the three effects discussed here.

Vacuum birefringence. A strong, slowly varying excitation (for example, a static magnetic-like twist background or a standing cavity mode) imprints a preferred direction in the otherwise isotropic vacuum medium. A weaker probe wave then sees two distinct normal modes: One polarized along the imprint and one across it. Because the quartic terms let the background modulate the local electromagnetic-like stiffness, the two modes acquire slightly different phase velocities—a birefringent split—even though no material is present. The model predicts that any persistent anisotropy seeded by strong twist, shear texture, or gentle scale gradients turns the vacuum into a weak, field-dependent birefringent medium.

Light-by-light scattering. Wave packets in this framework are self-propagating energy oscillations between fields. When two high-energy wave packets overlap, their combined intensity can slightly alter the local properties of the medium in the overlap region, creating a transient refractive grating. The waves then exchange energy and momentum through this self-induced grating—classically, a tiny four-wave–mixing process; in particle language, photon–photon scattering [21]. No charges or matter are required; the structured vacuum itself mediates the interaction. The effect scales with field intensity, overlap volume, and frequency, and is therefore exceptionally small for typical laboratory fields—but it is not forbidden.

Spin–quadrupole gravitational radiation. Because all vacuum sectors share the same geometric volume element, intense twist fields carry stress-energy and thus gravitate. When that stress-energy varies in time with a quadrupolar pattern, it sources shear (spin-2) waves even in the absence of moving masses. A particularly clean way to create such a source is optical spin flow: Crossed or counter-propagating circularly polarized beams, or a high-Q standing mode, produce a time-varying pattern of electromagnetic spin density with quadrupolar symmetry. The shear sector responds by emitting gravitational waves at twice the optical carrier frequency—hence "spin–quadrupole." The amplitude is tiny but, in principle, accumulates coherently with stored energy, frequency, and interaction volume, making engineered cavities natural targets for exploration. the effective medium without introducing matter.

The predictions above provide viability claims grounded in the structure of the quartic action of the structured vacuum medium. Exploration of these predictions is deferred to future work. The simulator described in Section 6 can serve as a bridge, numerically validating the regimes where the analytic approximations hold.

6 Simulating the Structured Vacuum

It turns out the structured vacuum model lends itself exceptionally well to simulation. It is trivially parallel and uses operations that modern libraries and GPUs are optimized to perform. We have implemented a simulator that can evolve structured vacuum fields on a cubic lattice with periodic boundary conditions (or, optionally, boundary damping). We use it to evaluate how the emergent behavior of the structured vacuum model can host physical objects like massive hedgehog particles and self-propagating solitons.

We model a discretized structured vacuum, where each point in the lattice is modeled as a `Site` object with a `u` field for the displacement vector u_i, an `e` field for the orthonormalized frame ${e^a}_\mu$, and a `sigma` field for the scale field σ_i.

Aside from the parameters that define the lattice size and site spacing, the simulator only uses the parameters presented in the action in this manuscript—no fine-tuning parameters are required. The simulator is written in C++ and uses several modern libraries for performing numerical operations. We have released the simulator as free and open source software, available at: `https://github.com/justinmeza/vacuum`.

We set the parameters in our simulator based on the coupling constants required to recover the various physical theories we have demonstrated throughout this manuscript. Table 2 shows the parameters used in our 64^3 structured-vacuum benchmark simulation, along with their physical provenance.

The simulator uses a fourth-order Runge-Kutta method to evolve the fields forward in time. The Runge-Kutta method is well-suited for this type of simulation, as it provides a good balance between accuracy and computational efficiency. A typical simulation run consists of an setup phase, where the fields are initialized, a Courant–Friedrichs–Lewy condition check to ensure stability, followed by a series of time steps where the fields are evolved according to the structured vacuum equations of motion. Status updates are printed to the console along with a hysteresis-based estimated time until simulation completion, and simulation snapshots are saved to storage for later analysis and visualization.

On a multi-core CPU, the simulator can take advantage of parallelism by using the OpenMP library to parallelize the evolution of the lattice sites. When using a GPU, the simulator can take advantage of parallelism to significantly speed up the simulation. We marshal the lattice sites in to and out of flat arrays of GPU memory, allowing us to perform vectorized operations on the fields. The user can set environment variables to control the number of threads used for OpenMP and/or interfacing with the GPU.

We provide a variety of visualization tools to analyze the simulation results. The tool kit includes Python scripts that can read the simulation output and plot 2D/3D heat maps of the fields, as well as animations of the simulation evolution over time. We also provide lower-level visualization tools that allow the user to visualize the raw contents of the fields in 3D. All of the figures shown in this section were generated in a few command lines using these tools.

We evaluate our simulator on a 64^3 structured vacuum lattice with periodic boundaries. The lattice spacing is set to 1.0 in natural units, and the simulation runs for 1000 steps. We run the simulator on a Framework 13 laptop with the specifications shown in Table 1. We evaluate two different types of excitations to stress test opposite dynamical behavior in the structured vacuum: charge-carrying solitons and propagating waves.

Processor	Intel® Core™ Ultra 7 165H
Core Configuration	16 cores (6P + 8E + 2LP-E), 22 threads
Base Frequency	1.5 GHz (E-cores), 2.0 GHz (P-cores)
Max Turbo	Up to 5.0 GHz (single P-core)
Graphics	Intel Arc™ GPU + Xe LPG
Memory	64 GB DDR5-5600 (dual channel)
Storage	8 TB PCIe Gen4 NVMe SSD
Operating System	GNU/Linux (Fedora 42, x86_64)

Table 1: Simulation system specifications.

6.1 Charge-Carrying Solitons

To seed a single charged soliton in the structured vacuum, we populate both dynamical fields on the lattice. Not only do both fields provide the soliton with mass and charge, but each also crucially balances the forces of the opposite field, allowing the soliton to oscillate in place without propagating.

For the displacement field, u_i, we initialize a purely azimuthal vortex that falls off as $u_\psi \propto \frac{1}{\rho^2}$. This displacement vortex supplies a long-range Coulomb-like field and stabilises the core. The rotation field, $e^a{}_\mu$, is the product of two smooth rotations about the radial unit vector $\hat{n}$ and stores the spin-texture and the torsion monopole that gives rise to electric charge.

Displacement Field We define the soliton core radius as R_e. Outside the core ($\rho > R_e$), we add a thin azimuthal halo,

$$\mathbf{u}(\rho, \psi) \;=\; u_0 \frac{R_e^2}{\rho^2} \hat{e}_\psi, \qquad \hat{\psi} \;\equiv\; (-\sin\psi, \cos\psi, 0),$$

which supplies the $\frac{1}{\rho^2}$ elastic strain required by the lowest-order field equation and ensures that the lattice Coulomb energy matches the continuum value once the system is let free to relax.

Rotation Field For every lattice site, we compute the distance $\rho = |\mathbf{x} - \mathbf{x}_c|$ to the chosen center and the radial direction $\mathbf{n}$. We then apply two rotation matrices successively:

1. A hedgehog flip $R_{\text{hedg}}(\rho) = \exp(+\chi(\rho)\,\mathbf{n}\cdot\mathbf{J})$ with $\chi(\rho) = \pi\,\text{sech}(\frac{\rho}{R_e})$. This carries the tetrad smoothly from the identity at infinity to a 180° inversion at the origin and produces a net torsion charge $q = +1$.

2. A half-spin twist $R_{\text{spin}}(\rho) = \exp(+\theta(\rho)\,\mathbf{n}\cdot\mathbf{J})$ with $\theta(\rho) = \pi\left[1 - \tanh(\frac{\rho}{R_e})\right]$. This extra SU(2) half-turn endows the defect with the correct fermionic sign change under a 2π rotation.

The stored frame is the composition of these two rotations, $e = R_{\text{spin}}R_{\text{hedg}}$. Because both angles go to zero exponentially, e is exactly the vacuum frame outside a sphere of radius $\approx 3R_e$; no high-k noise is introduced.

Figure 2 on page 33 shows 3D renderings of the initial state of shear and twist sectors and a cross-section of the z-axis at $n = 32$ every 500 simulation steps.

6.2 Propagating Waves

We encode a vector potential akin to magnetism in the displacement field u_i and a torsion field akin to electromagnetism in the rotation field $e^a{}_\mu$. These two fields are then enveloped in a Gaussian wave-packet that travels in the $+x$ direction.

Envelope and Carrier We define a separable Gaussian with longitudinal coordinate $\xi \equiv x - x_c$ and transverse coordinates $\eta \equiv y - y_c$, $\zeta \equiv z - z_c$,

$$\text{env}(\xi,\eta,\zeta) \;=\; \exp\left(-\frac{\xi^2}{2\sigma_x^2} - \frac{\eta^2}{2\sigma_y^2} - \frac{\zeta^2}{2\sigma_z^2}\right),$$

and a carrier phase $\phi = k\xi$ with wave–number $k = \frac{2\pi}{\lambda}$ and frequency $\omega = ck, c \equiv \frac{\lambda_T}{\lambda S}$. Throughout we fix the propagation, electric and magnetic axes to

$$\hat{\mathbf{k}} \;=\; +\hat{x}, \qquad \hat{\mathbf{e}} \;=\; +\hat{y}, \qquad \hat{\mathbf{b}} \;=\; +\hat{z} \quad \left(\hat{\mathbf{b}} \;=\; \hat{\mathbf{k}}\times\hat{\mathbf{e}}\right).$$

Displacement Field The complementary shear component is taken along the magnetic axis,

$$\mathbf{u}(\mathbf{x}) \;=\; \frac{A_0}{k}\,\text{env}\sin k\xi\,\hat{\mathbf{b}}, \qquad \partial_\xi\mathbf{u} \;=\; \frac{A_0}{k}\left(-\frac{\xi}{\sigma_x^2}\,\text{env}\sin k\xi + k\,\text{env}\cos k\xi\right)\hat{\mathbf{b}},$$

and the canonical momentum is set to $p_i = -c\,\partial_\xi u_i$ so that the linearised shear equation $\partial_t u_i = p_i$ holds exactly at $t = 0$

Rotation Field A small-angle rotation about the rank-1 generator $\hat{\mathbf{e}} \otimes \hat{\mathbf{k}}^\top$ produces the twist sector. The rotation angle and its longitudinal derivative are

$$\theta(\xi, \eta, \zeta) \;=\; \frac{A_0}{k}\,\mathrm{env}\cos k\xi, \qquad \theta'_\xi \;=\; A_0\left(-\frac{\xi}{\sigma_x^2}\,\mathrm{env}\cos k\xi - k\,\mathrm{env}\sin k\xi\right).$$

Energy Balance Because (e, u) and (ω, p) are phased as above, the quadratic energy density

$$\mathcal{E} \;=\; \lambda_S\,\frac{1}{2}\left|\nabla u\right|^2 + \lambda_T\,\frac{1}{2}\left|\nabla T^{\hat{0}}\right|^2 + \frac{1}{2}\left(\left|p\right|^2 + \left|\omega\right|^2\right)$$

is positive definite, localised by the Gaussian envelope, and splits equally between electric and magnetic sectors, reproducing the continuum Maxwell result to $\mathcal{O}\big(A_0^2\big)$.

Similar to before, Figure 3 on page 34 shows 3D renderings of the initial state of shear and twist sectors and a cross-section of the z-axis at $n = 32$ every 500 simulation steps. Since the wave packet is symmetric about the $x-$ and y-axes, we show a slice of the twist field at $z = \frac{N}{2}$ as a representative view. The color gradient represents the displacement of the vector u_i and rotation of the frame e_i at each site, with higher contrast colors indicating larger displacements.

Parameter	Natural	Simulation	Provenance
λ_S	$1/(32\pi G)$	1.0	Normalises the spin-2 kinetic term; fixes the code light-cone once $\lambda_T = \lambda_1$.
λ_T	$= \lambda_1$	1.0	Ensures photons propagate at the same speed as gravitons ($c_T = c_S$).
λ_C	$\omega_{\rm BD}$	10^5	Brans–Dicke bound from Cassini tracking: $\omega_{\rm BD} \gtrsim 4 \times 10^4$ [16].
α_S	$1/(16\pi G) = 2\lambda_1$	2.0	Matches the cubic part of the teleparallel Einstein–Hilbert [2, 3] action.
α_T	$\dfrac{2\alpha^2}{45, m_e^4} \sim 10^{-14}\lambda_1$	10^{-14}	Euler–Heisenberg one-loop value for low-energy quantum electrodynamics [13]; kept tiny so that non-linear optics is perturbative.
α_C	$\sim!10^{-120} M_{\rm Pl}^4$	10^{-8}	Toy dark-energy scale: tiny but large enough to resolve numerically.
α_H	$\leq 10^{-6}\lambda_1$	10^{-6}	GW170817, GRB170817A arrival-time bound on $\lvert c_{\rm GW} - c_\gamma \rvert$ [1].
α_R	$\leq 10^{-8}\lambda_1$	10^{-8}	Pulsar-timing array limit on running G (strain–dilaton mixing).
α_P	$\leq 10^{-8}\lambda_1$	10^{-8}	Same order as α_Σ to avoid excessive σ–torsion coupling.

Table 2: Couplings used in the structured-vacuum benchmarks together with their physical provenance.

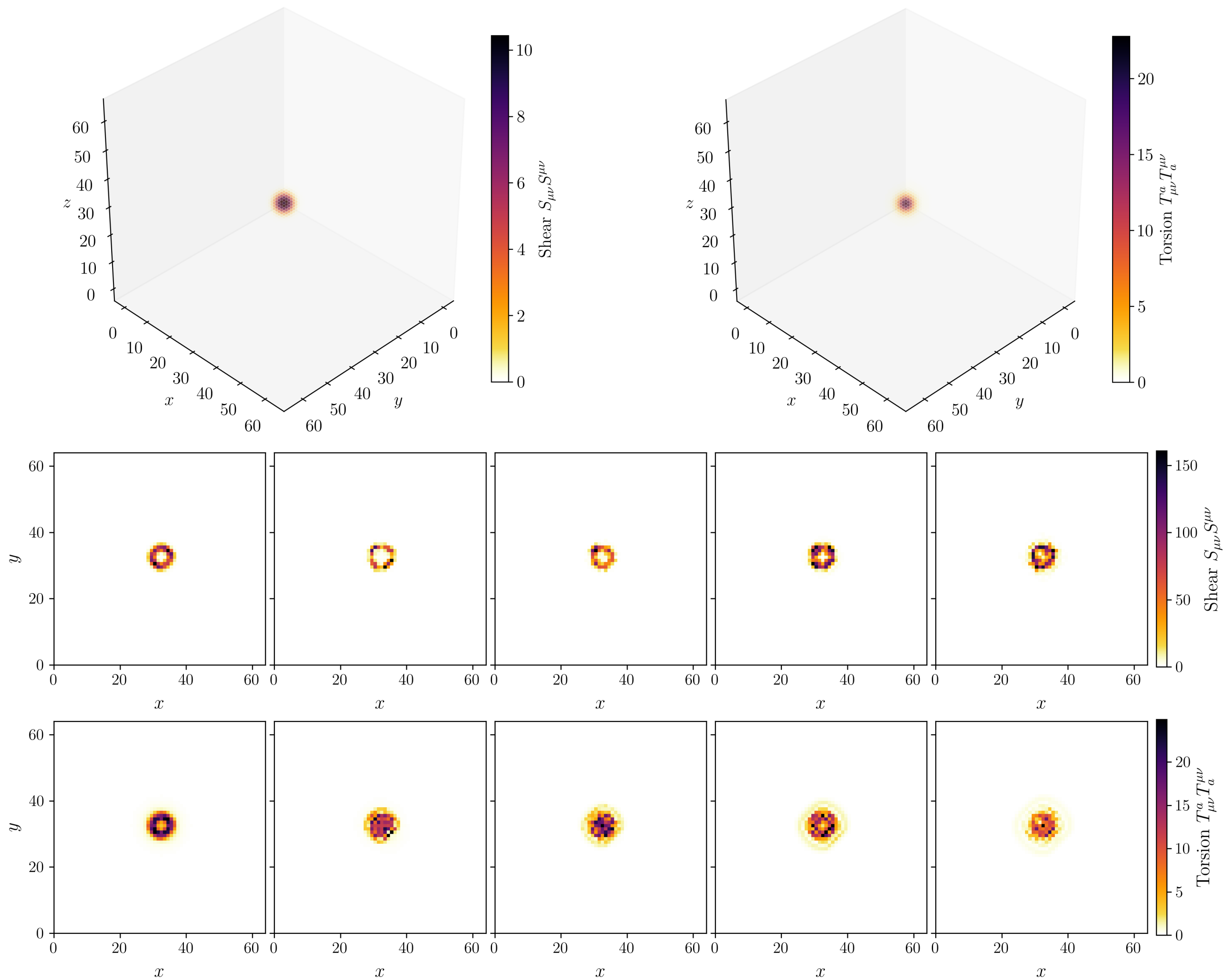


Figure 2: Simulated charge-carrying particle in the structured vacuum.

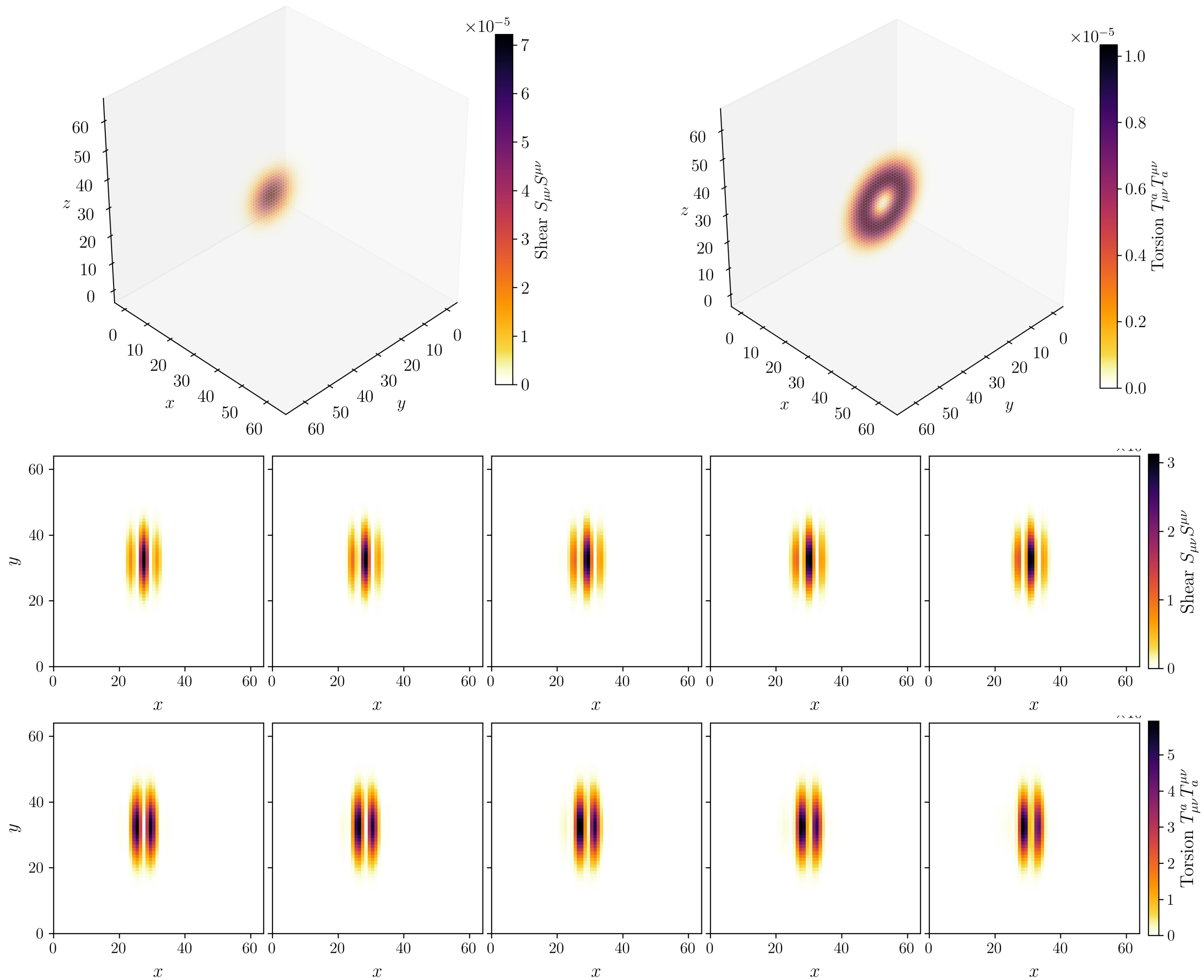


Figure 3: Simulated propagating wave in the structured vacuum.

7 Conclusions

In this work, we posed the question, *"What emergent behavior arises if the vacuum of space is not empty but has structure?"* To help answer this question, we endowed the vacuum with a geometric structure consisting of a displacement field, a rotation field, and a scale field. We then found that through successive derivation and elimination of unphysical interactions between these fields, our model naturally led to the emergence of relativistic gravity from its pure shear interactions, electromagnetism from its pure twist interactions, and a scalar-tensor field through its scale interactions. A potentially fruitful area of future work is to explore the cross-coupling between these fields—if they are measurable then they hint at hitherto undetected physics. We also release a free and open source simulator that allows for curious readers to explore the structured vacuum model and its emergent behavior in a numerical setting. It is our hope that this work encourages a curious and inquisitive view toward the vacuum of space and inspires further research into physics that may be hidden in seemingly empty space.

A Invariant Candidates

Below we enumerate every two- and four-derivative candidate that can be formed from the declared first-derivative blocks

$$S_{\mu\nu} \equiv \nabla_{(\mu}u_{\nu)}, \qquad T^a{}_{\mu\nu} \equiv 2\,\mathcal{D}_{[\mu}e^a{}_{\nu]}, \qquad \nabla_\mu\sigma,$$

together with $g_{\mu\nu}$, η_{ab}, $\epsilon_{\mu\nu\rho\sigma}$. We use the Hodge dual on spacetime indices, $\tilde{T}^a{}_{\mu\nu} \equiv \frac{1}{2}\,E_{\mu\nu}{}^{\rho\sigma}T^a{}_{\rho\sigma}$, with $E_{\mu\nu\rho\sigma} = \sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$.

For each discarded term we give the reason, using the criteria from Sections 4.1 and 5.1:

- [B] boundary / total divergence (integral independence)
- [0] identically zero (symmetry/antisymmetry)
- [E] proportional to a quadratic EOM (field redefinition)
- [P/T] violates the parity/time-reversal projection
- [R] redundant / algebraically dependent (Schouten identities, index symmetries, 2-form identities)
- [G] pure-*gauge* spin-connection artifact (we work with flat ω, $R[\omega] = 0$)
- [T] trace moved into σ (use $\sigma = \ln|e| \simeq S^\mu{}_\mu + \mathcal{O}(S^2)$ so isotropic dilation is carried by σ)

A.1 Quadratic (Two-Derivative) Invariants

The three kept terms are $S_{\mu\nu}S^{\mu\nu}$, $T^a{}_{\mu\nu}T_a{}^{\mu\nu}$, and $(\nabla\sigma)^2$. All other candidates fall into one of the cases below.

A.1.1 Terms linear in a first derivative

- $\nabla_\mu u^\mu = S^\mu{}_\mu$. $\boxed{\text{B}}$, integrates to a boundary; also $\boxed{\text{T}}$, isotropic part carried by σ.
- $\nabla_\mu\sigma$ alone. $\boxed{\text{B}}$, surface term.
- $T^a{}_{\mu\nu}$ alone. $\boxed{0}$, cannot form a scalar without ϵ (which would be $\boxed{\text{P/T}}$).

A.1.2 Built from ∂u

- Full gradients $\nabla_\mu u_\nu \nabla^\mu u^\nu$ and $\nabla_\mu u_\nu \nabla^\nu u^\mu$ decompose into $S_{\mu\nu}S^{\mu\nu} + A_{\mu\nu}A^{\mu\nu}$ with $A_{\mu\nu}$ antisymmetric. $A_{\mu\nu}A^{\mu\nu}$ is $\boxed{\text{R}}/\boxed{\text{G}}$; keep only $S_{\mu\nu}S^{\mu\nu}$.
- Trace terms such as $(S^\mu{}_\mu)^2$ or $S^\mu{}_\mu S_{\rho\sigma}S^{\rho\sigma}$ are $\boxed{\text{T}}$.

A.1.3 Built from $T^a{}_{\mu\nu}$

Let $T^\rho{}_{\mu\nu} \equiv e_a{}^\rho T^a{}_{\mu\nu}$. In 4D the classical invariants are

$$\mathcal{I}_1 = T_{\rho\mu\nu}T^{\rho\mu\nu}, \quad \mathcal{I}_2 = T_{\rho\mu\nu}T^{\mu\rho\nu}, \quad \mathcal{I}_3 = T_\mu T^\mu, \quad T_\mu \equiv T^\nu{}_{\mu\nu}.$$

- Keep $\mathcal{I}_1$ (i.e. $T^a{}_{\mu\nu}T_a{}^{\mu\nu}$).
- $\mathcal{I}_2, \mathcal{I}_3$ are $\boxed{\text{R}}$ and $\boxed{\text{E}}$.
- Parity-odd $T^a{\cdot}\tilde{T}_a = \epsilon^{\mu\nu\rho\sigma}T^a{}_{\mu\nu}T_{a\,\rho\sigma}/2$ is $\boxed{\text{P/T}}$ at quadratic order (its *square* appears at quartic order below and is kept).
- Nieh–Yan [11] density is $\boxed{\text{B}}/\boxed{\text{G}}$ for $R[\omega] = 0$.

A.1.4 Cross terms among $S, T, \nabla\sigma$

- $S_{\mu\nu}T^{a\,\mu\nu}$ is $\boxed{0}$.
- $S_{\mu\nu}\,\nabla^\mu\sigma\,\nabla^\nu\sigma$ has three derivatives, not quadratic.
- $T^a{}_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}\nabla_\rho\sigma\,\nabla_\sigma\sigma$ is $\boxed{\text{P/T}}$ and not quadratic.

A.1.5 σ-sector with second derivatives

- $(\Box\sigma)$ or $(\nabla_\mu\nabla_\nu\sigma)$ at quadratic order are $\boxed{\text{E}}/\boxed{\text{R}}$ via parts and $\Box\sigma = 0$.

A.2 Quartic (Four-Derivative) Invariants

The kept basis in Eq. (12) is:

$$\text{(i) } \alpha_S \left(S_{\mu\nu}S^{\nu\rho}S_{\rho\sigma}S^{\sigma\mu} - S_{\mu\nu}S^{\mu\nu}S_{\rho\sigma}S^{\rho\sigma}\right),$$

$$\text{(ii) } \tfrac{1}{4}\,\alpha_T \left(T^a\cdot T_a\right)^2, \qquad \text{(iii) } \tfrac{1}{4}\,\alpha_B \left(T^a\cdot\tilde{T}_a\right)^2,$$

$$\text{(iv) } \alpha_C \left((\nabla\sigma)^2\right)^2,$$

$$\text{(v) } \alpha_H\, S^{\mu\nu}S_{\mu\nu}\, T^a{}_{\rho\sigma}T_a{}^{\rho\sigma},$$

$$\text{(vi) } \alpha_R\, S^{\mu\nu}S_{\mu\nu}\,(\nabla\sigma)^2, \qquad \text{(vii) } \alpha_P\, T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\,(\nabla\sigma)^2.$$

A.2.1 Pure-shear S^4

Two non-trace contractions:

$$\mathcal{S}_1 = \mathrm{Tr}[S^4], \qquad \mathcal{S}_2 = (\mathrm{Tr}[S^2])^2.$$

Traceful variants are $\boxed{\text{T}}$. Keep $\mathcal{S}_1 - \mathcal{S}_2$. The orthogonal combination is $\boxed{\text{R}}$.

A.2.2 Pure-twist T^4

With $X^a_{\mu\nu} = T^a{}_{\mu\nu}$:

$$(X^a\cdot X_a)^2, \quad (X^a\cdot\widetilde{X}_a)^2, \quad (X^a\cdot X_a)\,(X^b\cdot\widetilde{X}_b).$$

Keep $(X^a\cdot X_a)^2$ *and* $(X^a\cdot\widetilde{X}_a)^2$ (both are parity even). The mixed product with a single dual, $(X^a\cdot X_a)\,(X^b\cdot\widetilde{X}_b)$, is $\boxed{\text{P/T}}$. Index-shuffle quartics are $\boxed{\text{R}}$.

A.2.3 Pure-scale $(\nabla\sigma)^4$

Only $\left((\nabla\sigma)^2\right)^2$ is independent. Second-derivative forms are $\boxed{\text{E}}/\boxed{\text{R}}$.

A.2.4 Shear–twist S^2T^2

$$\mathcal{H}_1 = S_{\mu\nu}S_{\rho\sigma}T^{a\,\mu\rho}T_a{}^{\nu\sigma}, \quad \mathcal{H}_2 = S_\mu{}^\rho S_{\nu\rho}T^{a\,\mu\sigma}T_a{}^\nu{}_\sigma,$$

$$\mathcal{H}_3 = S_{\mu\nu}S_{\rho\sigma}T^{a\,\mu\nu}T_a{}^{\rho\sigma}, \quad \mathcal{H}_4 = S_{\mu\nu}S_{\rho\sigma}T^{a\,\mu\rho}\widetilde{T}_a{}^{\nu\sigma}.$$

$\mathcal{H}_3$ is $\boxed{0}$, $\mathcal{H}_4$ is $\boxed{\text{P/T}}$ (one dual). $\mathcal{H}_{1,2}$ reduce to the kept helixity $S^{\mu\nu}S_{\mu\nu}\,T^2$ up to boundary terms ($\boxed{\text{R}}/\boxed{\text{E}}$).

A.2.5 Shear–scale $S^2(\nabla\sigma)^2$

$\mathcal{R}_1 = S_{\mu\nu}S^{\mu\nu}\,(\nabla\sigma)^2$ (kept), $\mathcal{R}_2 = S_\mu{}^\rho S_{\nu\rho}\,\nabla^\mu\sigma\,\nabla^\nu\sigma$ is $\boxed{\text{E}}/\boxed{\text{R}}$.

A.2.6 Twist–scale $T^2(\nabla\sigma)^2$

$\mathcal{P}_1 = T^a{}_{\rho\sigma}T_a{}^{\rho\sigma}\,(\nabla\sigma)^2$ (kept), $\mathcal{P}_2 = T^a{}_{\mu\rho}T_{a\,\nu}{}^\rho\,\nabla^\mu\sigma\,\nabla^\nu\sigma$ is $\boxed{\text{E}}/\boxed{\text{R}}$.

A.2.7 Mixed with ϵ

Single-dual (single ϵ) structures such as $S_{\mu\nu}S_{\rho\sigma}T^{a\,\mu\rho}\widetilde{T}_a{}^{\nu\sigma}$ or $(T^a\cdot\widetilde{T}_a)\,S^2$ are $\boxed{\text{P/T}}$. The *double-dual* parity-even combination $\big(T^a\cdot\widetilde{T}_a\big)^2$ is **kept** and listed above.

A.2.8 Quartics with derivatives on the blocks

Forms like $\nabla_\alpha S_{\mu\nu}\nabla^\alpha S^{\mu\nu}$, $\nabla_\alpha T^a{}_{\mu\nu}\nabla^\alpha T_a{}^{\mu\nu}$, $\nabla^\mu\sigma\,\nabla_\mu\big((\nabla\sigma)^2\big)$ are $\boxed{\text{R}}/\boxed{\text{E}}$.

A.2.9 "Cross-sector squares" of vanishing/odd quadratics

Squares of quadratic terms that are $\boxed{0}$ or violate $\boxed{\text{P/T}}$ remain excluded, *except* when the square restores even parity by introducing a second dual; this is precisely the case of $(T^a\cdot\widetilde{T}_a)^2$, which is included.

In summary, at two derivatives the only nontrivial scalars are S^2, T^2, and $(\nabla\sigma)^2$. At four derivatives, up to boundaries and quadratic EOMs, the complete parity-even, local, Lorentz-covariant, index-algebra-unique set built from these blocks is exactly the **seven** terms kept in Eq. (12).

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